Which Logic for the Radical Anti-Realist?

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Abstract

Since the ground-breaking contributions of M. Dummett (Dummett 1978), it is widely recognized that anti-realist principles have a critical impact on the choice of logic. Dummett argued that classical logic does not satisfy the requirements of such principles but that *intuitionistic logic* does. Some philosophers have adopted a more radical stance and argued for a more important departure from classical logic on the basis of similar intuitions. In particular, J. Dubucs and M. Marion (**?**) and (Dubucs 2002) have recently argued that a proper understanding of anti-realism should lead us to the so-called *substructural logics* (see (Restall 2000)) and especially linear logic. The aim of this paper is to scrutinize this proposal. We will raise two kinds of issues for the radical anti-realist. First, we will stress the fact that it is hard to live without structural rules. Second, we will argue that, from an anti-realist perspective, there is currently no satisfactory justification to the shift to substructural logics.

Introduction

One of the most striking outcomes of the controversy between semantic realism and semantic anti-realism concerns logic: it is widely held that an anti-realist position should result in a *revisionist* attitude with respect to classical logic. More precisely, according to the seminal contributions of M.

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Dummett, a coherent anti-realist should prefer intuitionistic logic to classical logic. Therefore, one would expect that different forms of anti-realism would result in different forms of revisionism. Recently, it has been argued by J. Dubucs and M. Marion (?) that an anti-realism more radical than the usual one justifies another logic than the intuitionistic, namely *linear logic* (Girard 1987). If one calls "moderate" the version of anti-realism advocated by Dummett, the landscape is going to be the following:

realism	$\Rightarrow$	classical logic
moderate anti-realism	$\Rightarrow$	intuitionistic logic
radical anti-realism	$\Rightarrow$	linear logic

The aim of this paper is not to take a position on the left-hand side of the tabular. We shall take position neither in the realism/anti-realism debate nor in the moderate/radical anti-realism debate. Our focus is the idea that a strengthening of the moderate anti-realist's basic insights leads to linear logic rather than to intuitionistic logic. We shall ask whether there is a path from the bottom-left cell to the bottom-right cell that is parallel to the usual path that goes from the middle-left cell to the middle-right one.

We will proceed as follows. In section 1, we give a rough reconstruction the basic tenets of anti-realism radical anti-realism. We then scrutinize both the consequences of committing oneself to substructural revisionism and the principles that could back up this commitment. In section 2, we argue that, because of the splitting of connectives, it is not easy to live without structural rules. Therefore the justification for such a shift has to be pretty firm. But in section 3, we show that there is currently no satisfactory foundation for substructural revisionism. In section 4, nonetheless, we briefly sketch a possible, game-theoretic, way to achieve such a foundation.

1 From anti-realism to substructural logic

1.1 Moderate anti-realism

We shall first reconstruct briefly the position of *moderate* anti-realism. Though we do not want to enter into an exegetical discussion, the view we present here could be called Dummettian anti-realism. We take moderate anti-realism to consist in two basic components: the anti-realist component *per se* and the revisionist component.

Moderate anti-realism starts with a rejection of truth-conditional semantics. According to truth-conditional semantics, the meaning of a declarative sentence S is the condition under which it is true – and to grasp the meaning of a sentence S is to grasp its truth-conditions. Furthermore, the truth or falsity of S is independent of our means of knowing it[1]: nothing precludes that the conditions under which S is true cannot be recognized as such when they obtain. To put it another way, S could be true even though it is not possible to know that it is true. According to the realist, truth in not epistemically constrained.

The anti-realist rejects precisely this lack of epistemic constraint: if a sentence S is true, then it should be possible to recognize that it is true. (This is the so-called "Knowability Principle".) There are two main arguments in favor of the Knowability Principle: if knowledge of meaning is to be analyzed as knowledge of truth-conditions, one has to be able to gain such knowledge (this is the *learnability argument*) and to manifest that one possesses it (this is the *manifestability argument*). As long as truth-conditions are recognition-transcendent, knowledge of truth-conditions does not satisfy these requirements. The realist thus fails to account for our mastery of language.[2]

As a consequence, the anti-realist rejects the notion of truth-condition as an adequate basis for a theory of meaning, and puts forward as an alternative the "conditions under which we acknowledge the statement as conclusively established" (Dummett 1978), p.226, or, as it is sometimes put, assertibility-conditions, *i.e.* the conditions under which one is justified to assert the sentence. When it comes to mathematical discourse, one is justified to assert a sentence just in case one has a proof of that sentence. Therefore, the meaning of a mathematical sentence consists in its provability conditions (as opposed to its mysterious recognition-transcendent truth-conditions).

This is it for our reminder of the basic tenets of moderate anti-realism. What we want to stress now is that (and how) *these tenets lead to logical revisionism*: they give us strong reasons to reject classical logic and, at least as far as mathematical discourse is concerned, to endorse intuitionistic logic.

[1] See (Dummett 1978), "Realism", p. 146: "Realism I characterize as the belief that statements of the disputed class possess an objective truth-value, independently of our means of knowing it..."

[2] In the case of mathematical discourse, see (Dummett 1978) : "If to know the meaning of a mathematical statement is to grasp its use; if we learn the meaning by learning its use, and our knowledge of its meaning is a knowledge which we must be capable of manifesting by the use we make of it: then the notion of truth, considered as a feature which each mathematical statement determinately possesses or determinately lacks, independently of ours means of recognizing its truth-value, cannot be the central notion for a theory of the meanings of mathematical statements [...]" (p. 225).

As a matter of fact, the path from anti-realism to logical revisionism is not as clear as one might wish. Actually, we think that there are *two ways* from anti-realism to logical revisionism: one that goes directly from the rejection of realism to the rejection of the law of excluded middle; and one that goes through proof-theoretic arguments from the endorsement of anti-realism to a thorough justification of intuitionistic logic[3] We shall call the first way *"high-level revisionism"* and the second *"low-level revisionism"*.

Let us elaborate on this distinction, which will play an important role in our discussion of radical anti-realism. *High-level revisionism* consists in rejecting the excluded middle on account of the Knowability Principle. More precisely, for Dummett, the rejection of the law of excluded middle (LEM for short) stems from the rejection of the principle of bivalence according to which every (meaningful, non-vague and non-ambiguous) declarative sentence is determinately true or false. Bivalence is not equivalent to LEM, but, as Dummett puts it, "once we have lost any reason to assume every statement to be either true or false, we have no reason, either, to maintain the law of excluded middle" ((Dummett 1991), p.9)

The exact argument for the rejection of bivalence is a matter of controversy[4]. J. Salerno has convincingly argued that Dummett's and Wright's (see Wright 1992, p. 43) arguments are unsound, but he has also proposed an amended version. His point is that the following three are incompatible:

(i) It is known that LEM holds.

(ii) The Knowability Principle is known: we know that for all A, if A is true, then it is possible to prove that A.

(iii) We do not know that for all A, either it is possible to prove A or it is possible to prove the negation of A. (This is a principle of epistemic modesty.)

[3]See (Read 1995) for a closely related presentation of the anti-realist case for revisionism: what we call "high-level revisionism" corresponds roughly to what Read calls the "Linguistic Argument" and what we call "low-level revisionism" corresponds to what he calls the "Logical Argument". The distinction is implicit in other places: e.g. in Tennant's (Tennant 1997), chapters 6-7 focus on "high-level revisionism", whereas chapter 10 focuses on "low-level revisionism".

[4]The question whether Dummettian anti-realism succeeds in vindicating logical revisionism has been (and is) much disputed. See C. Wright, "Anti-realism and revisionism" in (Wright 1993), (Tennant 1997), (Salerno 2000), (Cogburn 2002), (Cogburn 2003). In particular, (Tennant 1997) argues that Dummett's manifestation argument, even if it is an "attempted reductio of the principle of bivalence", "*is so far as it is directed against bivalence*, is, when properly regimented, revealed as embodying "a non-sequitur of numbing grossness"."

We have just briefly recalled the arguments in favor of the Knowability Principle. The principle of epistemic modesty is reansonable as well: there are large classes of sentences for which we do not possess any decision procedure, where by a decision procedure, we mean an effective method yielding a proof of A if A holds and a proof of the negation of A, if the negation of A holds. It follows then that we should reject the first claim, namely that we are entitled to assert LEM in full generality.

On the contrary, it is clear that LEM does hold for those classes of sentences for which we do possess a method for deciding them.[5] But the argument above shows that it is not sound in general to assume LEM, and that we should hold on to it *only when we are concerned with decidable classes of sentences*. The anti-realist is therefore a logical revisionist in so far as she draws a line between those statements for which LEM can be asserted and those for which it cannot. And decidability is the criterion used to draw this line, because decidability is both necessary and sufficient for us to be entitled to assert LEM.

Without entering into the details of Salerno's argument, we shall be content with this presentation of high-level revisionism. Let us consider now what we have called *low-level revisionism*. There is a normative component that any theory of meaning based on assertibility conditions should abide by: What can be inferred from a given sentence should not go beyond what is required in order to be entitled to assert it. This principle of *harmony* takes a precise form in the setting of natural deduction which is used to provide the meaning of logical and mathematical expressions.[6] In natural deduction, the assertability conditions – the conditions for being in position of asserting a statement – are given by the introduction rules, and the corresponding "exploitability conditions" – what can be inferred from a statement – are given by the elimination rules. The principle of harmony has it that every detour consisting in an introduction rule followed by an elimination rule for the same expression should be eliminable.

[5] A class of sentences is decidable if for any sentence in the class, a speaker is always in position to know whether she can assert it or not.

[6] Natural deduction can come either as a mono-conclusion system or as a multiple conclusion system. Harmony rules out classical logic only if the system admits only of a single conclusions at a time. Dummett has argued that using multiple conclusions is not ok because this presupposes an (unsound) classical understanding of disjunction. This point has been recently challenged by Restall (see (Restall 2005)). Restall proposes conceptual foundations for a system with multiple conclusions based on two primitives, assertion and denial. This seems to us to be a very promising response to the anti-realist challenge againt classical logic, though a discussion of Restall's arguments would lead us beyond the scope of this paper.

It turns out that the rules of intuitionistic logic satisfy harmony. But, under the assumption that the calculus should not allow for multiple conclusions, LEM or other principles yielding classical logic such as double negation elimination cannot be added in such a way that harmony obtains. We should therefore reject classical logic in favor intuitionistic logic, because the latter, but not the former, is satisfactory from a normative perspective. Low-level revisionism is thus based on a proof-theoretic semantics. It is important to note that this is a two-stage path. First, one endorses an assertibilty-conditions theory of meaning. Then, as a by-product, classical logic is disqualified and intuitionistic logic is justified.

A striking feature of logical revisionism along the lines of moderate anti-realism emerges when one compares high-level revisionism with low-level revisionism: both lead to the very same conclusion, namely that LEM should be rejected. On the one hand, High-level revisionism discards the principle of bivalence, leaving us with no reason to accept LEM. On the other hand, low-level revisionism justifies intuitionistic logic, which may be construed as classical logic minus LEM. Not only low-level revisionism is consistent with high-level revisionism, but it does not advocate any further departure from classical logic than the one which is required by high-level revisionism. There is thus some kind of "meta-harmony" betweeen the two levels of revisionism. As Dummett puts it ((Dummett 1993), p.75), "A theory of meaning in terms of verification is bound to yield a notion of truth for which bivalence fails to hold for many sentences which we are unreflectively disposed to interpret in a realistic manner". Low-level revisionism shows that a theory of meaning in terms of verification does yield the logic it is bound to yield on account of high-level revisionism.

1.2 Radical anti-realism

The radical anti-realist shares the basic tenets of the anti-realist, but she thinks her colleague is too shy when it comes to putting epistemic constraints on truth. As a consequence, the radical anti-realist will be a revisionist too, but she will be an even more radical one. We will start by explaining how and why the basic principles of anti-realism are radicalized, and then we will examine the consequence of this move for logic.

1.2.1 Decidability in principle and decidability in practice

According to the anti-realist, if a statement is true, one has to be able to recognize that it is true. And for LEM to hold, statements have to be decid-

able. But what does it mean to say that one has to be able to recognize that something is true, or to decide if a statement is provable or disprovable? On the one hand, the cognitive abilities of a not that gifted sophomore are certainly not the absolute norm by which truth should be constrained. On the other hand, the limitless powers of the divine intellect are not a reasonable candidate either. If truth is only constrained by what God can do, and if God can do anything, this is a cheap constraint indeed.

What are the norms by which recognizability of truth are to measured? The moderate anti-realism does not choose God's point of view; indeed Dummett acknowledges that if these norms were taken to be those of God, realism and anti-realism would conflate into one and the same position. However, moderate anti-realism is still quite *liberal* with respect to these epistemic constraints: for a set of sentences to be a decidable class, it is only required that such sentences might be decidable *in principle* by a creature with a finite mind, that is by finitary mechanical procedures.

Now, the problem is that moderate anti-realism has to face some kind of revenge. If truth has to be epistemically constrained in order to satisfy manifestability requirements, these constraints have to be strong enough to guarantee that knowledge of truth is manifestable. But think of a set of sentences which is decidable in principle, but such that the truth or falsity of some sentences can only be established by methods which are *practically* out of reach. In that case, what is there to be exhibited? If the decision procedure cannot actually be used and applied, in which sense would knowledge of these methods be any more human than God's knowledge? What would it mean to manifest such a knowledge, or to be able to acquire it? Thus it seems that for such a set of sentences the moderate anti-realist fails to satisfy the requirements that she has herself advertised against realism. Granting this point, moderate anti-realism appears as an *unstable* position: epistemic constraints on truth might be discarded right at the beginning, but if there are such constraints, they should be taken seriously and they should be measured by decidability *in practice* instead of decidability *in principle*.

1.2.2 The radical anti-realist crush on substructural logic

There have been various attempts to implement this radicalization, among which strict finitism is one of the most famous (as elaborated for example in (Wright 1993)). In this paper however, we shall focus on another version of radical anti-realism recently advocated in (Dubucs & Marion 2003).[7]

[7] We shall not explain here in any detail why we favor this approach over strict finitism. Basically, we agree with the arguments by Dubucs and Marion against strict finitism. Specifying by brute force what it means to be feasible – say it means "being doable in

According to Dubucs and Marion, the outcome of the radical anti-realist revision procedure should no longer be intuitionistic logic. They claim that substructural logics, and more precisely linear logic, are more faithful to the basic insights of anti-realism than intuitionistic logic.

Let us see why. In standard presentations of sequent calculus, different types of rules are distinguished. There are on the one hand the *logical rules*, which make for the introduction of logical connectives, and there are on the other hand the *structural rules*, like the rules of Weakening and Contraction, which correspond to properties of the consequence relation itself.[8] Here are Weakening and Contraction:

$$\frac{\Gamma \vdash \Delta}{A, \Gamma \vdash \Delta}\ \text{Weakening (left)} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A}\ \text{Weakening (right)}$$

$$\frac{A, A, \Gamma \vdash \Delta}{A, \Gamma \vdash \Delta}\ \text{Contraction (left)} \qquad \frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A}\ \text{Contraction (right)}$$

The radical anti-realist's idea is that some substructural rules contain crucial elements of epistemic idealization. Hence, in order to "unidealize" logic from an epistemic point of view, one should control these structural rules. A new logical revisionism follows: the claim is now that a substructural logic like linear logic is justified from an anti-realistic point of view.

This is the radical anti-realist crush on substructural logic, and the aim of this paper is to evaluate it. One may basically evaluate such a proposal from two points of view: from the point of view of the *principles* that could lead to it and from the point of the *consequences* that would result from its endorsement. We will proceed to the evaluation from both points of view and deal with the following two questions:

- How can one live without structural rules?
- Why should one divorce from them?

less than n steps of computation", or "doable in a reasonably small number of steps" – is bound to lead to soritic paradoxes. Despite the criticisms that we develop on here, we take the proposal by Dubucs and Marion to be the most attractive one among various versions of radical anti-realism, precisely because it aims at getting a non-stipulatory grip on feasibility.

[8]Note that this distinction is not tied to the adoption of sequent calculus as a proof system. A similar point could be made using, say, natural deduction, tableaux methods or a dialogical setting. Arguably, any good framework for proofs is able to distinguish between abstract properties of the consequence relation, that may or may not be used in proofs, and the mere characterization of logical connectives by logical rules.

2 Life without structural rules

Opponents to semantic anti-realism have always been prompt to notice that there is something paradoxical in the anti-realist's position. The anti-realist bases his rejection of realism on slogans such as "meaning is use" and she ends up with a proposal to revise usage. Stated in polemical terms, this amounts to saying that the anti-realist's attitude towards use is opportunist. She invokes use when it is useful to do so and repudiates it when needed. Anti-realists grant the existence of such a tension, but claim that there is nothing preposterous in it. However, M. Dummett admits that the greater the revisions, the less plausible the theory, because "the principal purpose of a theory of meaning is to explain existing practice rather than to criticize it."[9]

Obviously, this tension will be all the more vivid in the context of radical anti-realism. If its advocate grants with Dummett that an increase in departure from "existing linguistic practice" yields a decrease in the theory's plausibility, then she cannot but hope that the shift to substructural logic is not a too dramatic revision.

How should we assess the acceptability of a revisionist proposal with respect to standard use? In the case of logical connectives, we take it that the most elementary inferences that speakers accept as part of a characterization of what these connectives mean should be recognized as valid. Of course what these inferences are is a matter of debate, but we shall argue that, on any account of what are the basic meaning-constitutive inferences for the logical connectives, the revision in point is quite severe. Our concern is related to a well-known feature of substructural logics, namely the so-called phenomenon of splitting of logical connectives. Let us consider two pairs of rules for conjunction in (intuitionistic) sequent calculus:

Introduction on the left	Introduction on the right	
$\dfrac{\Gamma, A \vdash C}{\Gamma, A \wedge B \vdash C} \quad \dfrac{\Gamma, B \vdash C}{\Gamma, A \wedge B \vdash C}$	$\dfrac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}$	additive $\wedge$
$\dfrac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C}$	$\dfrac{\Gamma_1 \vdash A \qquad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \wedge B}$	multiplicative $\wedge$

One can check that the two pairs of rules are equivalent, in the sense that each one can be derived from the other. But this derivation resorts crucially

[9] (Dummett 1993), "What is a Theory of Meaning? (II)", p.75.

to the structural rules of Weakening and Contraction. Without such rules, the equivalence does not hold. Therefore, in a context in which the structural rules are not valid, one gets two different conjunctive connectives: one that corresponds to the first pair of rules, and the other that corresponds to the second pair of rules. The latter is called *fusion* in the relevant logic literature, *multiplicative conjunction* or *times* (notation: $\otimes$) in the linear logic literature. The former is called *additive conjunction* or *with* (notation: &) in the linear logic literature.

The two right-introduction rules make the difference between the two conjunctions salient. In the additive case, there is one antecedent Γ which is common to $\Gamma \vdash A$ and $\Gamma \vdash B$, whereas in the multiplicative case, the antecedents may be different. For our discussion, the main question is to know what are the connections between these two connectives and our pre-theoretical notion of conjunction. Let us consider the two connectives in turn:

(i) $\otimes$: one can easily show that the following sequent is derivable:

$$A, B \vdash A \otimes B$$

which seems to be a highly desirable feature for a conjunction: to get A and B, I just need both A and B. Furthermore, the interaction between $\otimes$ and $\rightarrow$ satisfies the so-called residuation property:

$$A \rightarrow (B \rightarrow C) \equiv (A \otimes B) \rightarrow C$$

If A implies that B implies C, then I can get C from A and B, and *vice versa*. This is quite quite reasonable. But note that sequents of the form

$$A \otimes B \vdash A$$

are *not* derivable. From a pre-theoretical point of view, this behavior of $\otimes$ is weird. If I can show that A and B, why should not I be able to assert A?

(ii) &: the additive conjunction has welcome features as well. In particular, the following sequent is derivable:

$$A\&B \vdash A$$

But it is no longer possible to derive:

$$A, B \vdash A\&B$$

To put it bluntly, multiplicative conjunction seems to describe nicely the conditions under which a conjunction can be asserted but not what can be inferred from a conjunction. On the contrary, additive conjunction seems to describe nicely what can be inferred from a conjunction but not the conditions under which a conjunction can be asserted.

Several replies are available to the radical anti-realist. She can argue that one of two connectives is the true one. However, given what we have just said, this does not seem very plausible. Another reply would be to bite the bullet and consider that linear logic refines on our pre-theoretic use of conjunction which is ambiguous. From this point of view, contrary to what the layman thinks, there is no single well-defined notion of conjunction. The layman might be wrong, in the sense that our best theory of meaning might have among its consequences that "and" is indeed ambiguous. However, note that "and" fails the standard linguistic test for ambiguity, namely cross-linguistic disambiguation. "Bank" in English is ambiguous between some place where I can get money and some slop beside a river where I can sit and wait for the fish to bite the hook. One good reason to think that there are two different lexical entries for "bank" is that in other languages, like French, there are two different words for that, namely "banque" and "rive". We do not know of any spoken language in which there would be two different words for "and", one corresponding to additive conjunction, the other one to multiplicative conjunction.

Whatever the reply the radical anti-realist chooses, we take this to show that life without structural rules is not easy. Arguably, this does not constitute a knockdown argument to reject the radical anti-realist's proposal. But considering these difficulties, she better have very good reasons to divorce from structural rules. In other words, the reasons for rejecting structural rules have to be pretty strong in order to balance the cost of living without them. Hence, we now turn to the assessment of these reasons.

3 The anti-realist justification of substructural logic

Let us scrutinize more closely reasons given by radical anti-realists for dropping some of the structural rules. The way we proceed will follow our recon-

struction of moderate anti-realism. We will first consider high-level revisionism, and then low-level revisionism.

3.1 High-level revisionism

Radical anti-realism ensues from a strengthening of the epistemic constraints. What does high-level revisionism amount to in this context? A striking feature of the version of radical anti-realism we are discussing is that its denial of moderate anti-realism's idealizations leads to the rejection of *new* logical laws (the structural rules, instead of just LEM). Something here is puzzling. Requiring decidability in practice makes the class of problematic sentences larger, but why should such a shift have revisionist implications of a *different* kind?

Our point is the following. As we have stressed in the first section, the disagreement between realists and moderate anti-realists concerns classes of undecidable sentences: the moderate anti-realist rejects the disputed logical principle, namely LEM, precisely for those classes. Let us assume that there is an argument Π which relies on the principle that truth should be epistemically constrained and which does show that, for undecidable classes, LEM does not hold (Π is the kind of argument that we have mentioned in section 2). Let us assume furthermore that, in the previous principle, decidability in principle should be replaced by decidability in practice. As a consequence, Π is likely to be turned into a stronger argument Π', which shows that, for domains which are undecidable in practice, LEM does not hold. What is crucial here is that the shift from Π to Π' does not change the logical law (*i.e.* LEM) that is under dispute, but changes the scope of the domain of validity of that law. Arguably, the domain of validity of LEM becomes more restricted: the law is no longer valid for every domain[10] which is decidable in principle, but only for domains which are decidable in practice. Domains which are decidable in principle but not in practice will fall outside of the scope of the law.

The point has actually been made by C. Wright in his book on strict finitism:

> "...whereas the intuitionist is content to regard as determinately true or false any arithmetical statement whose truth value can be effectively computed, at least "in principle", the strict finitist will

[10]By a decidable domain, we mean a domain such that for any tuples of objects in the domain, for any predicate, a speaker is always in position to know whether the predicate applies to the objects. This is in keeping with our previous use of decidable as a property of classes of sentences.

> insist that the principle of Bivalence is acceptable only for statement the verification or falsification of which can be guaranteed to be humanly feasible." (in (Wright 1993), p.108)

As a consequence, the landscape to be drawn should not be:

realism	$\Rightarrow$	classical logic
moderate anti-realism	$\Rightarrow$	intuitionnistic logic
radical anti-realism	$\Rightarrow$	linear logic

but rather:

realism	$\Rightarrow$	LEM for all domains
moderate anti-realism	$\Rightarrow$	LEM restricted to domains decidable in principle
radical anti-realism	$\Rightarrow$	LEM restricted to domains decidable in practice

Restricting LEM to decidable domains and choosing intuitionistic logic is perfectly coherent. Since LEM is not valid in full generality, one should choose a logic such as intuitionistic in which the principle is not a theorem. It just happens that for some special domains, the decidable ones, LEM can be used, because of the property these domains have. The same is not true with restricting LEM to a subclass of decidable domains and choosing linear logic: the shift from intuitionistic logic to linear logic cannot be analyzed as a consequence of further restricting the validity of LEM.

To sum up, it is clear that, from a high-level perspective, radical anti-realism is bound to yield an even more radical revision. Nonetheless, it is not yet clear why the nature of this revision should be any different from the one advocated by the moderate anti-realist. Therefore, if the radical anti-realist is to propose a new kind of logic, such as linear logic, this justification has to take place from a low-level perspective. And, in any case, the convergence between high-level and low-level revisionism, which was a nice feature of moderate anti-realism, will be lost.

3.2 Low-level revisionism

Now we shall turn to low-level revisionism. The question is: can radical anti-realism do for linear logic what moderate anti-realism does for intuitionist logic? That is, can moderate anti-realism both vindicate the rules of a substructural logic and provide reasons to reject stronger systems?

To answer this question, some preliminary remarks are in order. First, moderate anti-realists do put forward a criterion, the criterion of harmony[11], which discriminates between acceptable and unacceptable pairs of rules. Radical anti-realists have to provide an analogous but more demanding criterion. Second, the radical anti-realist and the moderate anti-realist do not seem at first sight to talk about the same thing. The moderate anti-realist focuses on logical rules *stricto sensu*[12], whereas the radical anti-realist targets structural rules. Harmony is tailor-made for logical rules in a natural deduction format.

Thus in order to provide a complete justification, the moderate and the radical anti-realist have to propose admissibility criteria both for structural and for logical rules. Our aim will be to sketch the ways in which these expectations could be fulfilled. Four criteria are needed, as can be seen in the following tabular:

	moderate anti-realism	radical anti-realism
logical criterion	harmony	?
structural criterion	?	?

Let us consider first the admissibility criteria for logical rules (upper line). Right now, there is only one cell whose content is obvious. The moderate anti-realist takes the principle of harmony as a requirement on logical rules. It is clear that whatever is required for the moderate anti-realist is also required for the radical anti-realist. So the radical anti-realist's logical admissibility criterion has to be at least as strong as the principle of harmony.

But there is a question concerning the means which are available in order to eliminate the detours. For the moderate anti-realist, structural rules are available. But the radical anti-realist rejects the structural rules: therefore, when she requires harmony, she will also require that detours can be eliminated *without resorting to structural rules.*

For example, consider the following rules for conjunction:

$$\frac{\Gamma \vdash A \qquad \Gamma' \vdash B}{\Gamma, \Gamma' \vdash A \wedge B}\ \wedge\text{-intro} \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}\ \wedge\text{-elim} \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}\ \wedge\text{-elim}$$

These rules are harmonious as far as the moderate anti-realist is concerned.

[11] See paragraph 1.1 above.

[12] To our knowledge, Dummett does not discuss the validity of structural rules at all. One contingent reason might be that he uses systems of natural deduction in which structural rules are built-in rather than introduced as genuine rules.

Consider a detour consisting in an introduction rule followed by an elimination rule on the same conjunction:

$$\dfrac{\dfrac{\Gamma \vdash A \qquad \Gamma' \vdash B}{\Gamma, \Gamma' \vdash A \wedge B}\,\wedge\text{-intro}}{\Gamma, \Gamma' \vdash A}\,\wedge\text{-elim}$$

The detour can be eliminated and replaced by one application of Weakening:

$$\dfrac{\Gamma \vdash A}{\Gamma, \Gamma' \vdash A}\,\mathrm{W}$$

Either the anti-realist is able to eliminate all such detours without resorting to Weakening,[13] or she has to choose so-called 'pure' pairs of introduction and elimination rules for which the detours can be eliminated without structural rules. This is the case of the rules for additive conjunction or multiplicative conjunction. But mixing the two would not work.

To sum up, the most natural logical criterion for the radical anti-realist is nothing but a strengthened version of harmony, in which the use of structural rules is banned. The following picture arises:

	moderate anti-realism	radical anti-realism
logical criterion	harmony	strong harmony
structural criterion	?	?

Let us consider normative criteria for structural rules. What could the moderate anti-realist say? To start with, it is important to note that no analog of the principle of harmony is at hand. Roughly, harmony is meant to show that "nothing new" is introduced, in so far as harmony implies, at least in appropriate contexts, conservativity. But structural rules *do* introduce some new proof means: there are things which can be proved with structural rules which cannot be proved withou them. For example, if we consider a given atomic basis B in the sense of Prawitz (*i.e.* a set of mono-conclusion sequents containing only atomic sentences), it is in general possible that there is a sequent S which is not in B and which can be proven from B by using Weakening or other structural rules.

This means that one has to provide a full-fledged justification of structural rules, which does not rely on some sort of eliminability arguments. We suggest the following principle:

[13] It is actually hard to see how this could be done.

Preservation of Effectivity

A structural rule of the form
$\dfrac{\Gamma \vdash A}{\Gamma' \vdash A}$
is admissible iff,

if there exists an effective means to transform justifications for all sentences of Γ into a justification for A, then there exists an effective means to transform justifications for all sentences of Γ' into a justification for A'

The principle of preservation of effectivity is in the spirit of the BHK interpretation of logical constants. It is applied here at the meta-level to the consequence relation represented by the turnstile. Because of the close connection between the consequence relation in the meta-language and implication in the object-language, it comes as no surprise that our principle mirrors the BHK clause for implication. The anti-realist demands that proofs provide us with effective justifications, nothing less, but nothing more. Therefore, the principle of preservation of effectivity seems to express both necessary and sufficient conditions for the admissibility of structural rules.

This principle validates the standard structural rules. Weakening is admissible:

$$\frac{\Gamma \vdash A}{\Gamma, B \vdash A}$$

If one has an effective method to get a justification for A from justifications for sentences in Γ, one has also an effective method to get a justification for A from these justifications plus a justification for B. One just has to *discard* the unnecessary justification for B.

Contraction is admissible as well:

$$\frac{\Gamma, A, A \vdash A}{\Gamma, A \vdash A}$$

If one has an effective method to get a justification for A from justifications for sentences in Γ plus a justification for A and another one for the same sentence A, one has also an effective method to get a justification for A from the justifications for sentences in Γ and the remaining justification for A. One just has to *duplicate* the remaining justification for A whenever needed. (It is crucial here that the effective method provided for the upper sequent has to work whatever justifications for A are given.)

Exchange is admissible as well:

$$\frac{\Gamma, A, B, \Gamma' \vdash A}{\Gamma, B, A, \Gamma' \vdash A}$$

If one has an effective method to get a justification for A from justifications for sentences in Γ, for A, for B and for sentences in Γ', one has also an effective method to get a justification for A from justifications for sentences in Γ, for B, for A and for sentences in Γ'. One just has to look for the required justifications in the right place: the order on the left hand side of the sequent does not matter.

By contrast, let us have a quick look at the following rule, which we might call *stronk*:

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash B}\, stronk$$

stronk is some kind of strengthening which exhibits the same misbehavior as *tonk*. Let us assume that we have an effective procedure to get a justification for B from justifications for sentences in Γ and a justification for A. It might be the case that such a procedure makes an essential use of the justification provided for A, and that there is no effective procedure giving us a justification for B on the basis of Γ alone. Hence, *stronk* is not an admissible rule.

Preservation of effectivity vindicates the conservative attitude of moderate anti-realism towards structural rules. Our point in introducing this principle has only been to show that the gap left by the overlooking of structural rules could be bridged, and to prepare the ground for the discussion to come concerning radical anti-realism. We thus get the following picture:

	moderate anti-realism	radical anti-realism
logical criterion	harmony	strong harmony
structural criterion	preservation of effectivity	?

Let us turn now to the crucial last part of the discussion, namely the radical anti-realist view of structural rules. More precisely, we will discuss in turn three ways of filling the last blank in our tabular:

(i) Token preservation

(ii) Preservation of local feasibility

(iii) Preservation of global feasibility

(i) As noted above, the crucial claim of the radical anti-realist is that one should reject *both* the Weakening rule and the Contraction rule. By contrast, the radical anti-realist has no quarrel with the Exchange rule.[14] We will start by proposing a criterion which is designed to account for exactly this attitude towards structural rules, and then see whether it can be justified from the anti-realist perspective. Here is the suggestion:

> **Principle of Token Preservation**: A structural rule of the form
> $$\frac{\Gamma \vdash A}{\Gamma' \vdash A}$$
> is admissible if, for every formula B, the number of tokens of B is the same in Γ and in Γ'.

It is easy to see that the principle of Token Preservation rules out the Weakening and Contraction rules. On the contrary, the Exchange rule is justified according to it. Of course, pathological *stronk* is ruled out as well.

The Principle of Token Preservation thus seems to mirror adequately the radical anti-realist attitude towards the different structural rules. Of course, this is not enough: adopting the principle just on the ground that it yields the desired result would be an entirely *ad hoc* move, if there were no justification of it on account of the basic tenets of radical anti-realism.

Here is an attempt at such a justification of the principle. Linear logicians sometimes motivate their logic by providing an informal semantics for their calculus in terms of *resources and resource consumption* (see (Girard 1995)). On this interpretation, types of formulas stand for types of resources and a sequent of the form $\Gamma \vdash A$ expresses the fact that one can get an object of type A from resources corresponding to the elements of Γ. In this perspective, the Contraction rule becomes problematic because it says, for instance, that if one may get something of type A from two resources of type B, then one may get something of type A with just one resource of type B. But being

[14] One might ask why this is so. What about a super-radical anti-realist who would dismiss Exchange as well as Weakening and Contraction? Logical systems exist which would fulfill the super-radical anti-realist dreams. Non-commutative linear logic is one of them. The super-radical anti-realist would have to argue that the order in which evidence is given is not neutral with respect to our ability to draw conclusions on the basis of that evidence. A super-radical stance on feasibility might support this view. Think of an agent being asked whether it is valid to infer A from $A, A_1, ..., A_{100000}$, as compared to being asked whether it is valid to infer A from $A_1, ..., A_{52227}, A, A_{52228}, ..., A_{100000}$, the protocol being such that all A_is for $1 \leq i \leq 100000$ are actually presented to the agent. It might be super-feasible to answer the first question but not the second.

able to buy a pack of Marlboros with two bucks does not guarantee that that one can buy a pack of Marlboros with just one buck. It seems to us that this "resource interpretation" is conceptually defective from an anti-realist perspective. The reason is the following one. The resource interpretation is based on a sort of causal reading of the turnstile, formulas standing for types of objects such that the consumption of some can result in the production of others. In terms of the previous example, my two bucks can be traded for a pack of cigarettes. But the radical anti-realist is concerned with epistemic constraints on speakers. It is fallacious to assimilate the two perspectives. Of course, inference steps have a cognitive cost, and it might well be the case that some inference steps have a cognitive cost significantly higher than some others, so that a radical anti-realist should be particularly reluctant to admit them in her favorite logic. But, nonetheless, cognitive resources are not on a par with consumption goods. A justification does not disappear when I use it to build another justification in the same way that buying a pack of cigarettes makes a dollar or two disappear out of my pocket. For this reason, we do not think that a rejection of structural rules can be based on the "resource interpretation" of linear logic.

(ii) As a consequence, the radical anti-realist has to look for another kind of structural criterion of admissibility. The most promising line of thought consists in radicalizing what we have called above Preservation of Effectivity. It is very natural to do so since the basic criticism that the radical anti-realist addresses to her moderate cousin is that one should require not only effectivity in principle but effectivity in practice or feasibility. Therefore, one might wish to consider the following twist on Preservation of Effectivity:

> **Preservation of Local Feasibility**
>
> A structural rule of the form
>
> $$\frac{\Gamma \vdash A}{\Gamma' \vdash A'}$$
>
> is admissible if, if there exists a feasible means to transform justifications for all sentences of Γ into a justification for A, there exists a feasible means to transform justifications for all sentences of Γ' into a justification for A'

If one accepts to consider this requirement as a consequence of the radical anti-realist's position, the crucial question is: does Preservation of Local Feasibility gives us a reason to reject the Weakening and Contraction rules?

Consider Weakening. Let us assume that we have an effective and *feasible* method to get a justification for A from justifications for sentences in Γ. What kind of method do we get for extracting a justification for A from these justifications plus a justification for B? The same as before, except that one has now to discard the unnecessary justification for B. Why on earth would this not be feasible? After all, the idea is just to do as if no new justification would have been given, and to stick to the good old feasible procedure, *even if* we could now try to use in non-feasible ways the new justification we have just been provided with. But note that for Preservation of Local Feasiblity to hold, it is sufficient that the property of being in possession of a feasible procedure is preserved. It would not make sense to require that all possible procedures are feasible, since it is clear that there might always be non-feasible ways of doing feasible things (try to unload a ton of sand from a truck with a pitchfork instead of a spade).

The same goes with Contraction. Let us assume that we have an effective method to get a justification for A from justifications for sentences in Γ plus a justification for B and another one for the same sentence B. As we said, to get a transformation procedure corresponding to the lower sequent, one has to be able to "re-use" the justification provided for B. Again, why on earth would this not be feasible? By assumption, the justification for B has to be "simple" enough to be dealt with in the transformation procedure. But if this is the case, why would it suddenly cease to be "simple" enough to be re-used?

(iii) The radical anti-realist might reply that it is intuitively clear that it is harder to get a proof of A from Γ and B than from Γ alone, because one has to take into consideration that B might be necessary to prove A from Γ. If we do not grant this point, the anti-realist might claim that our disagreement reflects a brute conflict of intuitions and that our arguments do not bear upon his analysis of what the logic of feasible proofs is. But by saying so, the radical anti-realist makes a significant shift from the question of the preservation of feasibility for transformation procedures to the question of the feasibility of establishing that, let's say, A follows from Γ. The latter question concerns the complexity of the consequence relation, *i.e.* a global property of the logic. Instead of discussing whether some property is locally preserved by applying Weakening or Contraction, we are now dealing with a different idea, namely that admissibility of structural rules should be judged on the basis of their effects on logical systems. This global requirement of feasibility bears upon the calculus as a whole, as opposed to the local requirement we had introduced. The new principle could be spelt in the following way:

Preservation of Global Feasibility

A set of structural rules S preserves global feasibility w.r.t. a set of logical rules L, iff, if $\vdash_L$ is feasible, then $\vdash_{L+S}$ is feasible as well.

For the radical anti-realist's intuitions to be mathematically vindicated, the complexity of the consequence relation of a logic without structural rules should be lower than the complexity of the consequence relation of the same logic to which structural rules have been added. In particular, the radical anti-realist seems to be committed to the claim that it is feasible to establish whether A follows from Γ in linear logic whereas it is not the case in intuitionistic logic.

However, at this point, some well-known results in computability theory and complexity theory preclude such vindication. As a matter of fact, the consequence relation is decidable and PSPACE-complete in intuitionistic logic[15] but is undecidable in full linear logic. If one drops the exponentials, linear logic becomes decidable but is still PSPACE-complete. Furthermore, if one shifts from full linear logic to affine logic, i.e. linear logic plus the Weakening rule, one goes from undecidability to decidability (see (Lincoln 1995)). As a consequence, there seems be no correlation between the feasibility of establishing that A is a consequence of Γ and the rejection of structural rules. To the contrary, such a rejection sometimes makes matter worse (see also (Rosset 2010), who makes a similar point against strict anti-realism).

4 A way out for radical anti-realism?

Up to now, our analysis of radical anti-realism has led to two main claims:

- Radical anti-realists have to provide a low-level justification of their choice of linear logic.
- Such a justification is still to be provided. In particular, three direct attempts have been examined and shown to fail.

Our criticism of the three putative requirements on structural rules (token preservation, preservation of local feasibility and preservation of global feasibility) are not on a par. For purely mathematical reasons, the idea of preserving global feasibility seems to us to be misguided. The problem with

[15] We hereby mean that the problem of deciding whether a pair of formulas stand in this relation or not is PSPACE-complete.

token preservation is its lack of conceptual support: the anti-realist does not explain why justifications should share the properties of consumption goods. In a sense, preservation of local feasibility can be construed as some kind of conceptual support in favor of token preservation. However, our criticism of local feasibility suggests that, on this account, an informal notion of justification is not likely to invalidate structural rules.

To be fair, the radical anti-realist could blame our failure to see why structural rules are problematic on our informal analysis of justifications. She could claim that, on her view of what justifications are, two justifications can be (substantially) better than one. Now, of course, such a view has to be spelled out. Some help could come from the proof semantics that have been given for linear logic: as long as such a semantics could be considered to provide a formal counterpart for a reasonable notion of justification, it would provide intuitive counterexamples to the admissibility of structural rules and to the provability of the sequents that can be derived by using them.

The game semantics which have first been proposed by Blass (Blass 1992) provide a case in point. In this setting, justifications are defined as winning strategies for a designated player on two-players games (the two players are P for Player and O for opponent, where P is the designated player who tries to "verify" the formula). Given games for atomic formulas, each complex formula is associated with a mathematically well-defined game, which is defined by recursion on the syntax. Provability of a sequent $\vdash A$ amounts to the existence of a winning strategy for P no matter what the atomic games are. Blass gives an example of an infinite game for an atom G such that O has a winning strategy for $G \otimes G$ even though she does not have one for G alone. Intuitively, this accounts for the fact that "G and G" (where "and" is multiplicative conjunction) can be harder for me to justify than "G" alone (my opponent might be able to refute my claim that G and G though she is not able to refute my claim that G).

However, a crucial feature of Blass' semantics is the use of infinite two-players game, which are responsible for the failure of determinacy and hence for the differences between G and $G \otimes G$. From an anti-realist perspective, the meaning of infinite "justificatory debates" is not clear. But there are other ways to lose determinacy. In particular, a natural anti-realist constraint on strategies would consist in feasibility requirements: a justification for a formula G should not be any kind of winning strategy, but a feasible one, where feasibility would be captured in terms of a measure of complexity on strategies (say, we should only consider strategies computable by finite automata of a given size to reflect the cognitive limitations of the agent, see (Neyman 1998) for more on this). In a given finite game, one of the two players has some winning strategies, but all these strategies may well fall

outside of the class of the feasible ones. Hence the failure of determinacy.

To put it bluntly, there is on the one hand a story told by the radical anti-realist explaining to us why we should worry about intuitionistic logic and its anti-realist foundations. On the other hand, there are various available semantics for linear logic which show for what kind of notions of justifications structural rules can fail to be admissible. A thorough vindication of linear logic by radical anti-realism would have to make this story and one of these semantics meet. Our point in the previous section was to suggest that this is by no way an easy task, and that an elaborate notion of justification is needed. Our suggestion in the present section is that game semantics together with complexity constraints on available strategies could be a reasonable candidate.[16]

Conclusion

As the name says, radical anti-realism is a radicalized version of anti-realism. In terms of conceptual motivations, it rests upon the idea that anti-realism stops halfway in its request of feasibility. Radical anti-realism demands feasibility in practice instead of mere feasibility in principle. In terms of logical revisionism, radical anti-realism advocates a more radical departure from classical logic. Its favored logical systems are substructural logics, and in particular linear logic. We have proposed a systematic discussion of the arguments from conceptual motivations to logical revisionism, comparing the case for the move from feasibility in practice to substructural logics to the case for the move from feasibility in principle to intuitionistic logic.

Only of the two main arguments available to moderate anti-realists is available to radical anti-realists. Direct objections to the law of excludded middle will not speak in favor of systems strictly weaker than intuitionistic logic. The other option is to lay down explicit admissibility criteria for logical and structural rules. Harmony is the famous criterion put forward by Dummett for logical rules. There is no such standard proof-theoretic criterion for structural rules. To make up for this, we have suggested three possible criteria that might appeal to radical anti-realists.

[16] As a witness of the recent interest of strict anti-realists in games, see (Marion 2005). In discussion, Greg Restall has suggested another possible line of defense for strict anti-realism. By the well-known Curry-Howard correspondence between proofs and programs, proofs in intuitionistic logic can be turned into computable functions represented by λ-terms. Dropping structural rules shrinks the class of typable λ-terms (Weakening allows for empty binding and Contraction for multiple binding). The strict anti-realist would have a point if he could show that the class of functions typable in linear logic corresponds to a more feasible class of functions than those typable in intuitionistic logic.

These criteria were based on a straightforward implementation of the idea of feasibility in practice in a broadly "BHK-like" framework. None of them seems to us to be satisfactory, so that one lesson of our efforts is that the move from radical anti-realism to substructural logics is probably more problematic than the corresponding move from anti-realism to intuitionistic logic. In the very last section of this paper, we suggest that a promising alternative would be to construe proof-theoretic semantics,[17] such as game semantics, as implictly providing a logical analysis of justifications on which feasibility requirements could be imposed. The next step, which falls outside the scope of this paper, obviously is to work out the details of such a proposal.

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[17] Philosophers sometimes mean by proof-theoretic semantics a semantics for mathematical sentences in terms of conditions of provability. Proof-theorists mean by proof-theoretic semantics semantic accounts of the nature of proofs (including criteria of identity for proofs).

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