Dependence in logic

Logical Constants Workshop
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Introduction

- Henkin quantifier: $\begin{pmatrix} \forall x \exists y \\ \forall z \exists w \end{pmatrix}$ (1959)
- Hintikka and Sandu's IF-logic: $\forall x \exists y \forall z \exists w/x$ (1989)
- Hodges' semantics for IF-logic: Using **sets of assignments**. (1997)
- Vännänen's Dependence Logic: Using dependence as **atomic** property: $=\!(x, y, z)$ (2007)

$$\forall x \exists y \forall z \exists w \, (=\!(z, w) \wedge \dots)$$

HODGES' SEMANTICS

- X is a **team**, i.e., a set of assignments.
- $M \models_X \varphi$.
- For first-order φ: $M \models_X \varphi$ iff for all $s \in X$, $M \models_s \varphi$.
- $M \models_X =\!(x, y, z)$ iff there is a function $f : M^2 \to M$ such that for every $s \in X$: $s(z) = f(s(x), s(y))$; or, equivalently:

$$M \models_X =\!(x, y, z)$$

iff for all $s, s' \in X$ if $s(x) = s'(x)$ and $s(y) = s'(y)$ then $s(z) = s'(z)$.

x	y	z
1	11	11
1	12	11
2	11	2
2	13	2

- $M \not\models_X x = z$
- $M \not\models_X x \neq z$
- $M \models_X =\!(x, z)$
- $M \not\models_X =\!(x, y)$

Hodges' semantics II

- $M \models_X \varphi \wedge \psi$ iff $M \models_X \varphi$ and $M \models_X \psi$.
- $M \models_X \varphi \vee \psi$ iff there are Y and Z such that $M \models_Y \varphi$, $M \models_Z \psi$ and $X = Y \cup Z$.
- $M \models_X \exists x \varphi$ iff there is $f \colon X \to M$ such that $M \models_{X[f/x]} \varphi$.
- $M \models_X \forall x \varphi$ iff $M \models_{X[M/x]} \varphi$.

$$X[f/x] = \{\, s[f(s)/x] \mid s \in X \,\}.$$

$$X[M/x] = \{\, s[a/x] \mid a \in M, s \in X \,\}.$$

Branching in natural languages

Some relative of each villager and some relative of each townsmen hate each other. (Hintikka 1974)

$$\begin{pmatrix} \forall x \exists y \\ \forall z \exists w \end{pmatrix} \big(V(x) \wedge T(z) \to (R(x, y) \wedge R(z, w) \wedge H(y, w)) \big)$$

Most of the dots and most of the stars are all connected by lines. (Barwise 1979)

Two examiners marked six scripts. (Davies 1989)

Branching as an operator

For monotone quantifiers the branching of Q_1 and Q_2 as in

$$\begin{pmatrix} Q_1 x \\ Q_2 y \end{pmatrix} R(x, y)$$

can be represented by the quantifier $\mathrm{Br}(Q_1, Q_2)$ as in $\mathrm{Br}(Q_1, Q_2)xy\, R(x, y)$, where

$\mathrm{Br}(Q_1, Q_2)$ is the quantifier

$$\{ R \mid \exists A \in Q_1, B \in Q_2, A \times B \subseteq R \} .$$

Branching in Dependence Logic

$$M \models \mathrm{Br}(\forall\exists, \forall\exists)\, xyzw\, R(x, y, z, w)$$

iff

$$M \models \forall x \exists y \forall z \exists w \left(=\!(z, w) \wedge R(x, y, z, w)\right)$$

What about generalized quantifiers?

$$M \models \mathrm{Br}(Q_1, Q_2)\, xy\, R(x, y)$$

iff

$$M \models Q_1 x\, Q_2 y \left(=\!(y) \wedge R(x, y)\right)$$

?

Generalized quantifiers in Dependence Logic

Lifting functions

The **Hodges space** of order ideals on the power set is

$$\mathcal{H}(A) = \mathcal{L}(\mathcal{P}(A)).$$

Given $h : \mathcal{P}(A) \to \mathcal{P}(B)$ we define the **lift**:

$$\mathcal{L}(h) : \mathcal{H}(A) \to \mathcal{H}(B), \mathscr{X} \mapsto \downarrow\{\, h(X) \mid X \in \mathscr{X} \,\},$$

where $\downarrow\mathscr{X}$ is the downward closure of $\mathscr{X}$, i.e.

$$\downarrow\mathscr{X} = \{\, X \mid \exists Y \in \mathscr{X}, X \subseteq Y \,\}.$$

Lifting quantifers

- Q a monotone type $\langle 1 \rangle$ quantifier.
- $Q : \mathcal{P}(M^{n+1}) \to \mathcal{P}(M^n)$
- $\mathcal{L}(Q) : \mathcal{H}(M^{n+1}) \to \mathcal{H}(M^n)$
- Gives truth conditions for Q in Hodges semantics:

$$M \vDash_X Qx\varphi \text{ iff there is } F : X \to Q \text{ such that } M \vDash_{X[F/x]} \varphi.$$

  where $X[F/x] = \{\ s[a/x] \mid a \in F(s)\ \}$.
- $\mathcal{L}(\exists)$ and $\mathcal{L}(\forall)$ give the same truth conditions for $\exists$ and $\forall$ as we had before.

Proposition

For formulas φ with Q, but without dependence atoms:

$$M \vDash_X \varphi \text{ iff for all } s \in X,\ M \vDash_s \varphi.$$

Quantifiers and dependence

If Q contains no singletons then $M \not\models_X Qx\,(=\!(x) \wedge \varphi)$.

$$M \models \mathrm{Br}(Q_1, Q_2)xy\,R(x,y)$$

iff

$$M \models Q_1 x\, Q_2 y\,\big(=\!(y) \wedge R(x,y)\big)$$

Multivalued Dependence

A course database

Course	Student	Credits	Year
LC1510	Svensson	7.5 hp	2010
LC1510	Johansson	7.5 hp	2011
LC1520	Svensson	15 hp	2011
LC1520	AnderssonJohansson	15 hp	2011

- $=(\text{Course}, \text{Credits})$
- It is not true that $=(\text{Course}, \text{Student})$.
- F^{Student} takes values for Course and Credits and gives a set of possible values for Student.
- $F^{\text{Student}}(\text{LC1510}, 7.5\text{ hp}) = \{\text{ Svensson}, \text{Johansson }\}$
- F^{Student} is determined by the value of Course.
- $[\text{Course} \twoheadrightarrow \text{Student}]$
- $[\twoheadrightarrow]$ **dependent** on context.
- $F^{\text{Student}}(\text{LC1510}, 7.5\text{ hp}, 2010) = \{\text{ Svensson }\}$
- $F^{\text{Student}}(\text{LC1510}, 7.5\text{ hp}, 2011) = \{\text{ Johansson }\}$
- $[\twoheadrightarrow]$ **not** closed downwards: **Not** true that $[\twoheadrightarrow \text{Student}]$

Multivalued dependence and teams

- If $s \in X$ then $F_X^y(s) = \{ a \mid s[a/y] \in X \}$.

Definition

$M \vDash_X [\bar{x} \twoheadrightarrow y]$ if F_X^y is determined by the values of $\bar{x}$. (Only for $y \notin \bar{x}$.)

Proposition

$M \vDash_X [\bar{x} \twoheadrightarrow y]$ iff for all $s, s' \in X$ such that $s(\bar{x}) = s'(\bar{x})$ there exists $s_0 \in X$ such that $s_0(\bar{x}) = s(\bar{x})$, $s_0(y) = s(y)$, and $s_0(\bar{z}) = s'(\bar{z})$, where $\bar{z}$ are the variables in $\mathrm{dom}(X) \setminus (\{ \bar{x} \} \cup \{ y \})$.

- $M \vDash_X [\bar{x} \twoheadrightarrow y]$ is **dependent on context** and **not closed downwards**.
- $M \vDash_X {=}(\bar{x}, y)$ iff $X \vDash [\bar{x} \twoheadrightarrow y]$ and F_X^y only takes singleton values.

Generalized quantifiers and multivalued dependence

Proposition

If Q is monotone then $M \vDash \mathrm{Br}(Q_1, Q_2)xy\, R(x, y)$ iff

$$M \vDash Q_1 x\, Q_2 y \left([\twoheadrightarrow y] \wedge R(x, y)\right).$$

Proposition

FOL + multivalued dependencies has the same strength, on the level of sentences, as ESO, and thus as Dependence Logic.

Proposition [Galliani -11]

The class of teams definable in FOL + multivalued dependencies are exactly the ones definable in ESO (with an extra predicate for the team).

Embedded multivalued dependence

- Multivalued dependence is axiomatizable (as an atomic property).
- Multivalued dependence is **dependent on context**.

Definition

$M \models_X [\bar{x} \twoheadrightarrow \bar{y} | \bar{z}]$ iff $Y \models [\bar{x} \twoheadrightarrow \bar{y}]$ where Y is the projection of X onto $\{ \bar{x}, \bar{y}, \bar{z} \}$.

- $[\bar{x} \twoheadrightarrow \bar{y} | \bar{z}]$ is **independent on context**.
- This is the **independence atom** introduced by Väänänen and Grädel: $\bar{y} \perp_{\bar{x}} \bar{z}$ iff $[\bar{x} \twoheadrightarrow \bar{y} | \bar{z}]$
- However, embedded multivalued dependence is **not** axiomatizable. [Sagiv Walecka 1982] (Which both functional dependence and multivalued dependence are.)
- Embedded multivalued dependence is definable in FOL with multivalued dependencies. [Galliani -11]

Thank you for your attention.