On the Constructive Meaning of Implication in Classical and Intuitionistic Logic

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Constructive semantics

Constructive semantics:
Meaning of a proposition is given in terms of what conditions must be fulfilled in order to assert the proposition.

If these conditions demand the possession of a proof, then a constructive semantics must describe

- what are proofs of atomic propositions,
- what are proofs of logically complex propositions.

Description usually given as an inductive definition.

We consider Prawitz's semantics (1971): intended for intuitionistic logic,

(... Sandqvist (2009): same ideas, but intended for classical logic.)

Problem:
Involves a conflation of the concepts of admissibility and derivability.

Prawitz's approach

Prawitz (1971) proposed a systematic account of the BHK interpretation:

By clauses that inductively establish what is a construction of a sentence over a Post system **S**.

Post systems **S** are given by production rules

$$\frac{p_1 \quad \dots \quad p_n}{p_{n+1}}$$

where the p_i are atomic sentences.

Definition of *constructions of sentences* (Prawitz)

(i) k is a construction of an atomic sentence A over **S** if and only if k is a derivation of A in **S**.

(ii) k is a construction of a sentence $A \supset B$ over **S** if and only if k is a constructive object of the type of $A \supset B$ and for each extension $\mathbf{S}'$ of **S** and for each construction k' of A over $\mathbf{S}'$, $k(k')$ is a construction of B over $\mathbf{S}'$.

Prawitz's approach

Idea:
When a construction of an implication is shown, it must remain for extensions of the underlying Post system.

The semantics validates

$$\dfrac{\begin{matrix}[A]\\ \vdots \\ B\end{matrix}}{A \supset B}\ (\supset \mathrm{I}) \qquad \text{and} \qquad \dfrac{A \qquad A \supset B}{B}\ (\supset \mathrm{E})$$

Claim: Prawitz's semantics is inadequate.

Peirce's law can be validated in the implicational fragment $\{\supset\}$ of minimal logic *NM* extended by Post systems.

Problem: Conflation of concepts of derivability and admissibility.

Admissibility versus derivability

In natural deduction, a rule is

- admissible when it is guaranteed that there is a formal proof of the conclusion if there are formal proofs of the premisses,
- derivable if there is a derivation in the system, having as open assumptions no more than the premisses of the rule and as endformula the conclusion of the rule.

Premisses of rules may depend on assumptions; addition of an admissible rule could extend the set of theorems:

For an admissible rule $\dfrac{(A \supset B) \supset A}{A}$ the non-theorem $((A \supset B) \supset A) \supset A$ would be provable by $(\supset \mathrm{I})$:

$$\dfrac{\dfrac{[(A \supset B) \supset A]}{A}}{((A \supset B) \supset A) \supset A}\ (\supset \mathrm{I})$$

Proviso:
An admissible but non-derivable rule can only be added if its application is restricted to premisses which do not depend on any assumptions.

Peirce's rule

Only the fragment $\{\supset\}$ of minimal logic NM has to be considered.

Peirce's law not provable in this fragment, hence Peirce's rule not derivable.

But Peirce's rule can be shown to be admissible for any Post system $\mathbf{S}$:

Theorem (Admissibility of Peirce's rule)

For all atomic A, if there is a closed derivation (i.e. a proof) of

$$(A \supset B) \supset A$$

in the fragment $\{\supset\}$ of NM over any Post system $\mathbf{S}$, then there is a closed derivation of

$$A$$

in this fragment.

Proof.

By a **construction** k transforming any proof of the premiss into a proof of the conclusion. □

Admissibility and derivability conflated

(1) For any Post system **S** Peirce's rule

$$\frac{(A \supset B) \supset A}{A}$$

is admissible for atomic A.

(2) Then by Prawitz's clause (ii) the construction k is a construction of Peirce's law

$$((A \supset B) \supset A) \supset A$$

for any **S** and atomic A.

(3) In the implicational fragment we then obtain Peirce's law in general.

But Peirce's law is not provable in this fragment of NM, since provability is defined by referring to the concept of derivability.

Admissibility and atomic rules discharging assumptions

Admissibility does **not** hold for atomic rules discharging assumptions.
Consider a system $\mathbf{S}^+$ with rules (adapted from Schroeder-Heister)

$$\dfrac{\begin{matrix}[0=0]\\ \vdots \\ 0=1\end{matrix}}{0=1}\,(*) \qquad \text{and} \qquad \dfrac{0=1}{0=0}\,(**)$$

In the fragment $\{\supset\}$ of NM extended by $\mathbf{S}^+$ we have:

$$\dfrac{\dfrac{\dfrac{\dfrac{[0=0]^1 \qquad [0=0 \supset 0=1]^2}{0=1}\,(\supset E)}{0=1}\,(*)^1}{0=0}\,(**)}{(0=0 \supset 0=1) \supset 0=0}\,(\supset I)^2$$

Derivation does not contain a proof of $0=0$.

Hence Peirce's rule would not be admissible in such a system: there would be a formal proof (of the instance $(0=0 \supset 0=1) \supset 0=0$) of its premiss without there being a formal proof (of the instance $0=0$) of its conclusion.

Incompleteness of minimal logic

Constructive validity of Peirce's law for atomic A for systems restricted to production rules would imply incompleteness of the implicational fragment of minimal logic *NM*, and therefore of *NM* itself.

Irrespective of whether the conflation is recognized, the validation of Peirce's law could be *prevented* by allowing for systems $\mathbf{S}^+$ of atomic rules *discharging assumptions*.

But:

Incompleteness of *NM* (and also of intuitionistic logic *NJ*) can nonetheless be shown when such systems $\mathbf{S}^+$ are allowed and if validity is still considered to be given by a notion like admissibility.

Incompleteness of minimal logic

Theorem (Admissibility of the Mints rule)

The Mints rule

$$\frac{(A \supset B) \supset (A \vee C)}{((A \supset B) \supset A) \vee ((A \supset B) \supset C)}$$

is admissible for any system $\mathbf{S}^+$ *of atomic rules discharging assumptions. More importantly, this can be shown constructively by proof-theoretical means.*

Proof.

We give a procedure showing how to obtain a proof of the conclusion, given a proof of the premiss in any system $\mathbf{S}^+$. Suppose there is a proof of the premiss $(A \supset B) \supset (A \vee C)$ in such a system $\mathbf{S}^+$. Then there is also a normal form proof of it in $\mathbf{S}^+$. The last rule application in this normal form proof is $(\supset \mathrm{I})$:

$$\cfrac{\begin{array}{c}[A \supset B]\\ \vdots \\ A \vee C\end{array}}{(A \supset B) \supset (A \vee C)}\ (\supset \mathrm{I})$$

Incompleteness of minimal logic

Proof. (cont'd)

Consider the subderivation:

$$\begin{array}{c} A \supset B \\ \vdots \\ A \vee C \end{array}$$

Last rule application cannot be an atomic rule. Three cases:

(1) If it is an elimination rule, then there is a major premiss elimination sequence (i.e. a path $\xi_1, \ldots, \xi_n, \xi_{n+1}$ where the formulas $\xi_1, \ldots, \xi_n$ are major premisses of elimination rules and ξ_{n+1} is either a major premiss or the endformula) starting with open assumption $A \supset B$ as major premiss of $(\supset \mathrm{E})$. Hence, there is a derivation of the minor premiss A of this $(\supset \mathrm{E})$ from (maybe) an open assumption $A \supset B$.

(2) If the last rule application is $(\vee \mathrm{I})$, then there is a derivation of A from $A \supset B$,

(3) or there is a derivation of C from $A \supset B$.

In any case, by an application of $(\supset \mathrm{I})$ followed by $(\vee \mathrm{I})$ we have a proof of the conclusion $((A \supset B) \supset A) \vee ((A \supset B) \supset C)$ in system $\mathbf{S}^+$. □

Incompleteness of minimal logic

Thus there is a construction, now for any system $\mathbf{S}^+$, that transforms a proof of the premiss into a proof of the conclusion.

The Mints rule is not derivable in *NM* or in *NJ*, and the formula

$$\underbrace{((A \supset B) \supset (A \vee C))}_{\text{premiss of Mints rule}} \supset \underbrace{(((A \supset B) \supset A) \vee ((A \supset B) \supset C))}_{\text{conclusion of Mints rule}}$$

is thus neither provable in *NM* nor in *NJ*.

Any attempt to define a constructive semantics for the fragment $\{\supset, \vee\}$ which uses Prawitz's clause (ii) for implication will imply incompleteness of *NM* and *NJ* if the semantics validates disjunction introduction and elimination.

Prawitz's semantics is intended to capture the BHK interpretation.
In this case the criticism carries over to BHK.

Alternative: Consider *NJ* constructively incomplete, and look for another way of defining a new constructive logic better suited.

Non-atomic extensions of NJ?

Consider non-atomic extension E of NJ:
$(p \supset q) \supset (p \vee r)$ only extralogical axiom, for some atomic sentences p, q, r.

Suppose π were a proof of $((p \supset q) \supset p) \vee ((p \supset q) \supset r)$ in E.

Either π uses the extralogical axiom or it does not.

(1) If it does, then the axiom is a top occurrence of π. Thus there would be a derivation π showing

$$(p \supset q) \supset (p \vee r) \vdash_{NJ} ((p \supset q) \supset p) \vee ((p \supset q) \supset r).$$

But this is impossible, because the Mints rule is not derivable in NJ.

(2) If π has no top occurrence of the axiom, there would be a proof of

$$((p \supset q) \supset p) \vee ((p \supset q) \supset r)$$

in NJ. But this is not even a classical law.

Hence Mints rule is not admissible in E.

Problem: Semantic characterization of logical constants cannot be given by an inductive definition.