The set-theoretic multiverse: a model-theoretic philosophy of set theory

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Set theory as Ontological Foundation

Set theorists commonly take their subject as constituting an ontological foundation for the rest of mathematics, in the sense that other abstract mathematical objects can be construed fundamentally as sets, and in this way, they regard the set-theoretical universe as the universe of all mathematics.

On this view, mathematical objects *are* sets. Being precise in mathematics amounts to specifying an object in set theory.

For example, a *function* is a set of ordered pairs with the functional property. A *group* is a set with a binary operation having certain features. Real numbers, ordinal numbers, and even natural numbers are construed as sets.
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The Set-Theoretical Universe

These sets accumulate transfinitely to form the universe of all sets. This cumulative universe is regarded by set theorists as the domain of all mathematics.

The orthodox view among set theorists thereby exhibits a two-fold realist or Platonist nature:

- First, mathematical objects exist as sets, and
- Second, these sets enjoy a real mathematical existence, accumulating to form the universe of all sets.

A principal task of set theory, on this view, is to discover the fundamental truths of this cumulative set-theoretical universe.
I emphasize that the orthodox view holds the set-theoretical universe to be unique—it is the universe of all sets. On this view, interesting set-theoretical questions, such as the Continuum Hypothesis and others, have definitive final answers in this universe. Set theorists aim to discover these answers, but how?

Proponents point to the increasingly stable body of regularity features flowing from the large cardinal hierarchy, such as the attractive determinacy consequences and the accompanying uniformization and Lebesgue measurability properties in the projective hierarchy of sets of reals. These indicate in broad strokes that we are on the right track towards the final answers to these set theoretical questions.
The Universe View

To summarize, let me describe as the *Universe view* the position holding that there is a universe of all sets, that all mathematical objects exist as sets in this universe, that it is unique, and that this universe exhibits certain set theoretical truths, such as the Zermelo-Frankael ZFC axioms and a substantial portion of the large cardinal hierarchy, together with consequences, such as Projective Determinacy and much more.

Meanwhile, I intend to defend a contrary view.
An opening thrust

I begin with an opening thrust against the Universe view.

A paradox for the Universe view. While adherents hold that there is a single universe of all sets, nevertheless the most powerful set-theoretical tools developed in the past half-century are most naturally understood as methods of constructing alternative set-theoretical universes.

forcing, ultrapowers, canonical inner models, etc.

Much of set theory over the past half-century has been about constructing as many different models of set theory as possible. These models are often made to exhibit precise, exacting features or to exhibit specific relationships with other models.
An abundance of universes

Set theorists have constructed an enormous variety of models of set theory. Would you like $\text{CH} + \neg \diamondsuit$? Do you want $2^{\aleph_n} = \aleph_{n+2}$ for all $n$? Or Suslin trees? We build such models to order.

As a result, the fundamental object of study in set theory has become the model of set theory. We have $L$, $L[0^\#]$, $L[\mu]$, $L[\tilde{E}]$; we have models $V$ with large cardinals, forcing extensions $V[G]$, ultrapowers $M$, cut-off universes $L_\delta$, $V_\alpha$, $H_\kappa$, universes $L(\mathbb{R})$, HOD, generic ultrapowers, boolean ultrapowers, etc. Forcing especially has led to a staggering variety of models.

Set theory has apparently discovered an entire cosmos of set-theoretical universes, connected by forcing or large cardinal embeddings, like lines in a constellation filling a dark night sky. Set theory now exhibits a category-theoretic nature.
Forcing → A staggering collection of Universes

Forcing is a set-theoretic method (Cohen, 1963) for constructing a larger model of set theory from a given model.

Begin with a ground model $V \models \text{ZFC}$ and poset $P \in V$. Adjoin an ideal “generic” element $G$, a $V$-generic filter $G \subseteq P$, and with it construct the forcing extension $V[G]$, akin to a field extension.

$$V \subseteq V[G]$$

Objects of $V[G]$ are constructible algebraically from objects in $V$ and the new object $G$. The ground model $V$ has a surprising degree of access to the objects and truths of $V[G]$.

Forcing has been used to construct a staggering variety of models of set theory.
The Multiverse View

These alternative universes pose difficulties for the Universe view, which must explain them as imaginary. A competing philosophical position accepts them as fully real.

**The Multiverse view.** The philosophical position holding that there are many set-theoretical universes.

This is a brand of *realism*, since the alternative set-theoretical universes have a full mathematical existence. The view in part is that our mathematical tools—forcing, etc.—have offered us glimpses into other mathematical worlds. These tools have provided evidence that there are other mathematical worlds.

A Platonist may recoil at first, but actually, this IS a kind of Platonism, namely, Platonism about universes, second-order realism. My thesis is that set theory is mature enough to adopt and analyze this view mathematically.
The Multiverse

The multiverse view places a focus on the connections *between* the various universes, such as the relations of ground model to forcing extension or model to ultrapower.

For example, early forcing results were usually stated as relative consistency results by starting with a model of $\varphi$ and providing $\psi$ in a forcing extension.

$$\text{Con}(\text{ZFC} + \varphi) \rightarrow \text{Con}(\text{ZFC} + \psi)$$

Contemporary work often states theorems as:

*If $\varphi$, then there is a forcing extension with $\psi$.*

Such a policy retains important information connecting the two models.
Do forcing extensions of the universe exist?

The central dispute is whether there are universes outside \( V \), taken under the Universe view to consist of all sets. A special case captures the essential debate:

**Question**

Do forcing extensions of the universe exist?

On the Universe view, forcing extensions of \( V \) are illusory. On the Multiverse view, \( V \) is an introduced constant, referring to the universe currently under consideration.

The central issue is the ontological status of the forcing extension \( V[G] \). Does forcing offer us a glimpse into alternative mathematical universes? Or is this an illusion?
How forcing works

Before proceeding, let’s say a bit more about how forcing works.

We start with a model $V$ of set theory and a partial order $\mathbb{P}$ in $V$.

**Suppose $G \subseteq \mathbb{P}$ is an $V$-generic filter**, meaning that $G$ contains members from every dense subset of $\mathbb{P}$ in $V$.

$$V \subseteq V[G]$$

The forcing extension $V[G]$ is obtained by closing under elementary set-building operations. Every object in $V[G]$ has a name in $V$ and is constructed directly from its name and $G$.

Remarkably, the forcing extension $V[G]$ is always a model of ZFC. But it can exhibit different set-theoretic truths in a way that can be precisely controlled by the choice of $\mathbb{P}$. 
Forcing over countable models

Traditionally, forcing was often used with countable transitive ground models.

One starts with a countable transitive model $M$ of set theory. Since $M$ is countable, it has only countably many dense sets, and so we easily construct an $M$-generic filter $G$ by diagonalization. Thus, we construct the forcing extension $M[G]$ very concretely.

Drawbacks:

- Metamathematical issues with existence of countable transitive models of ZFC.
- Limited to forcing over only some models.
- Pushes much of the analysis into the meta-theory.
Forcing over $V$

A more sophisticated approach to forcing develops the technique within $\text{ZFC}$, so that one can force over any model of set theory.

We want to make the move from forcing over a model $M$ to forcing over the universe $V$.

“Removing the training wheels”

There are a variety of ways to accomplish this.
Ontological status of generic filters

Those who take $V$ as the unique universe of all sets object:

"There are no $V$-generic filters"

Surely we all agree that for nontrivial forcing notions, there are no $V$-generic filters in $V$.

But isn’t the objection like saying:

"There is no square root of $-1$"

Of course, $\sqrt{-1}$ does not exist in the reals $\mathbb{R}$. One must go to a field extension, the complex numbers, to find it.

Similarly, one must go to the forcing extension $V[G]$ to find $G$. 
Imaginary objects

Historically, $\sqrt{-1}$ was viewed with suspicion, and existence was deemed imaginary, but useful.

Eventually, mathematicians realized how to simulate the complex numbers $a + bi \in \mathbb{C}$ inside the real numbers, representing them as pairs $(a, b)$ with a peculiar multiplication $(a, b) \cdot (c, d) = (ac - bd, ad + bc)$. This way, one gains some access to the complex numbers, or a simulation of them, from a world having only real numbers.

The case of forcing is similar. We don’t have the generic filter $G$ inside $V$, but we have various ways of simulating the forcing extension $V[G]$ inside $V$, using the Naturalist account of forcing, or via the Boolean-valued structure $V^B$ and its quotients.
Naturalist Account of Forcing

Suppose that $V$ is a universe of set theory and $\mathbb{P}$ is a notion of forcing. Then there is a class model of the theory expressing what it means to be the corresponding forcing extension.

Namely, the theory keeps everything true in $V$, relativized to a class predicate, and also asserts that there is a $V$-generic filter.

The language has $\in$, constant symbols for every element of $V$, a predicate for $V$, and constant symbol $G$. The theory asserts:

1. The full elementary diagram of $V$, relativized to predicate.
2. The assertion that $V$ is a transitive proper class.
3. $G$ is a $V$-generic ultrafilter on $\mathbb{P}$.
4. ZFC holds, and the (new) universe is $V[G]$.

In practice, this is how forcing arguments proceed

Naturalist Account

Another way to describe it is:
For any forcing notion $P$, the universe $V$ can be embedded

$$V \lessdot \overline{V} \subseteq \overline{V}[G]$$

into a class model $\overline{V}$ having a $\overline{V}$-generic filter $G \subseteq \overline{P}$.

The point here is that the entire model $\overline{V}[G]$ and the embedding exist as classes in $V$.

The map $V \lessdot \overline{V}$ is a Boolean ultrapower map.
Taking forcing at face value

The Multiverse position makes the straightforward interpretation that with forcing, we have discovered the existence of other mathematical universes.

We have some access to them via names and the forcing relation, but they exist outside our own universe.

“Like Galileo peering through the telescope...”

It is quite remarkable that distinct set concepts are closely enough related to be analyzed from the mathematical perspective of each other.

This gives rich substance to the multiverse perspective as a model-theoretic philosophy of set theory.
An analogy with Geometry

Over a century ago, Geometers were shocked to discover non-Euclidean geometries.

The first consistency arguments presented them as simulations within Euclidean geometry (e.g. ‘line’ = great circle on sphere).

In time, geometers accepted the alternative geometries more fully, with their own independent existence, and developed intuitions about what it is like to live inside them.

Today, geometers have a deep understanding of these alternative geometries.

- They reason about them externally, as embedded spaces.
- But also internally, using their new intuitions.
- They reason about them abstractly, using isometry groups.

No-one now regards the alternative geometries as illusory.
The Universe view is simulated inside the Multiverse by fixing a particular universe $V$ and declaring it to be the absolute set-theoretical background, restricting the multiverse to the worlds below $V$.

This in effect provides absolute background notions of countability, well-foundedness, etc., and there are no $V$-generic filters in the restricted multiverse...

From the Multiverse perspective, the Universe view appears to be just like this, restricting attention to a fixed lower cone in the multiverse.

Arguments about the final answers to set-theoretic questions amount to: which $V$ to fix?
Case study: the Continuum Hypothesis

The Continuum Hypothesis (CH) is the assertion that every set of reals is either countable or equinumerous with $\mathbb{R}$.

This was a major open question from the time of Cantor, and appeared at the top of Hilbert’s famous list of open problems in 1900.

The Continuum Hypothesis is now known to be neither provable nor refutable from the usual ZFC axioms of set theory.

Gödel proved that CH holds in the constructible universe $L$.

Cohen proved that every model $V \models ZFC$ has a forcing extension $V[G]$ with $\neg CH$. The generic filter directly adds any number of new real numbers, so that there could be $\aleph_2$ of them or more in $V[G]$, violating CH.
CH in the Multiverse

More important than mere independence, both CH and $\neg$CH are forceable over any model of set theory. Every $V$ has:

- $V[\vec{c}]$, collapsing no cardinals, such that $V[\vec{c}] \models \neg$CH.
- $V[G]$, adding no new reals, such that $V[G] \models$ CH.

That is, both CH and $\neg$CH are easily forceable. We can turn CH on and off like a lightswitch.

We have a deep understanding of how CH can hold and fail, densely in the multiverse, and we have a rich experience in the resulting models. The Multiverse view is that CH is settled by this knowledge.

In particular, I shall argue that the CH can no longer be settled in the manner that set theorists formerly hoped it might be.
The traditional template for settling CH

Set theorists traditionally hoped to settle CH this way:

**Step 1.** Produce a set-theoretic assertion $\Phi$ expressing a natural ‘obviously true’ set-theoretic principle.

**Step 2.** Prove that $\Phi$ determines CH.
   That is, prove that $\Phi \rightarrow CH$,
   or prove that $\Phi \rightarrow \neg CH$.

And so, CH would be settled, since everyone would accept $\Phi$
and its consequences.

I argue that this template is now unworkable. Because of our
rich experience and familiarity with models having CH and
$\neg CH$, either implication above immediately casts doubt on the
naturality of $\Phi$. So we cannot accept such a $\Phi$ as obviously true.
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Other attempts to settle CH

More sophisticated attempts to settle CH do not rely on this traditional template.

Woodin has advanced an argument for ¬CH based on Ω-logic, appealing to desirable structural properties of this logic.

What the Multiversist desires in such a line of reasoning is an explicit explanation of how our experience in the CH worlds was somehow illusory, as it seems it must be for the argument to succeed.

Since we have an informed, deep understanding of how it could be that CH holds, even in worlds close to any given ¬CH world, it will be difficult to regard these worlds as imaginary.
Multiverse Axioms

Somewhat more speculatively, I would like now to propose principles expressing a fuller version of the multiverse perspective.

This perspective amounts in many ways to a model-theoretic philosophy of set theory.

We have numerous set concepts, each leading to its set-theoretic universe. Often, these diverse universes are closely enough related that they can be analyzed from the perspective of each other.

We seek principles expressing the universe existence properties we expect to find in the multiverse.

I aim here to be a little provocative.
Naive multiverse background

The principal multiverse idea is that there should be a large collection of universes, each a model of (some kind of) set theory. We include models of $\text{ZFC}$, $\text{ZF}$, $\text{ZF}^-$, $\text{KP}$ and so on, perhaps even second-order number theory, as this is set-theoretic in a sense.

But we need not consider all universes in the multiverse equally, and we may simply be more interested in the parts of the multiverse consisting of universes satisfying very strong theories, such as $\text{ZFC}$ plus large cardinals.

There is little need to draw sharp boundaries, and we may easily regard some universes as more set-theoretic than others.
Multiverse as second-order Maximize

The Multiverse view is expansive.

- The multiverse is as big as we can imagine.
- At any time, we are living inside one of the universes.
- We do not expect individual worlds to access the whole multiverse.

Maddy used the MAXIMIZE principle—no undue limitations on set existence—to explain resistance to $V = L$ as an axiom. It seems unduly limiting that every set should be constructible.

Similarly, here, we follow a second-order version of MAXIMIZE: we seek to place no undue limitations on which universes exist in the multiverse.
Universe Existence Principles

Let’s begin with some basic multiverse principles.

### Realizability Axiom

For any universe $V$, if $W$ is definable in $V$ and a model of set theory, then $W$ is a universe.

This includes all definable inner models, such as $L$, $\text{HOD}$, $L(\mathbb{R})$, etc., but also ultrapowers, and so on.

### Forcing Extension Axiom

For any universe $V$ and any forcing notion $\mathbb{P}$, there is a forcing extension $V[G]$.

But we also expect non-forcing extensions.
Mirage of tallness

Every model of set theory imagines itself as very tall, containing all the ordinals.

But of course, a model contains only all of its own ordinals, and it can be mistaken about how high these reach, since other models may have many more ordinals.

We have a rich experience of this with models of set theory. Thus, I propose the following multiverse principles:

Top Extension Axiom

For every universe $V$, there is a much taller universe $W$ with $V \preceq W_{\theta}$. 
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Mirage of uncountability

Every model of set theory imagines itself as enormous, containing the reals \( \mathbb{R} \), all the ordinals, etc. It certainly believes itself to be uncountable.

But of course, we know that models can be mistaken about this. The Lowenheim-Skolem theorem show that there can be countable models of set theory, and these models do not realize there is a much larger universe surrounding them.

Countability is a notion that depends on the set-theoretic background. On the multiverse perspective, the notion of an absolute standard of countability falls away.

Countability Axiom

Every universe \( V \) is countable from the perspective of another, better universe \( W \).
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Absolute standard of Well-foundedness

Set-theorists have traditionally emphasized the well-founded models of set theory.

The concept of well-foundedness, however, depends on the set-theoretic background. Different models of set theory can disagree on whether a structure is well-founded.

Every model of set theory finds its own ordinals to be well-founded.

But we know that models can be incorrect about this. One model $W$ can be realized as ill-founded by another, better model.
Absolute standard of well-foundedness

This issue arises even with the natural numbers. Every model of set theory finds its own natural numbers $\mathbb{N}$ to be ‘the standard’ model of arithmetic.

But different models of set theory can have different (non-isomorphic) ‘standard’ models of arithmetic. The $\mathbb{N}$ of one model of set theory can be viewed as non-standard from another model.

We have a rich experience of this in set theory. Every set theoretic argument can take place in a model, in which the ordinals appear to be well-founded, but actually, the model is ill-founded with respect to another better model.
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The mirage of well-foundedness

The idea of an absolute standard of well-foundedness appears sensible, I agree, when there is a fixed absolute set-theoretic background, as with the Universe view.

But on the multiverse view, we lose this fixed set-theoretic background.

On the multiverse view, we seem to have no reason to support the idea of a fixed absolute notion of well-foundedness.

Well-foundedness Mirage Axiom

Every universe $V$ is ill-founded from the perspective of another, better universe.

In sum, every universe $V$ is a countable ill-founded model from the perspective of another, better universe.
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In sum, every universe $V$ is a countable ill-founded model from the perspective of another, better universe.
Reverse Ultrapowers

When we are living in a universe $V$ and have an ultrapower embedding $V \rightarrow M$, then we can iterate this map forwards

$$V \rightarrow M \rightarrow M_2 \rightarrow M_3 \rightarrow \cdots$$

The later models $M_n$ do not realize that their maps have already been iterated many times.

Reverse Embedding Axiom

For every universe $V$ and every embedding $j : V \rightarrow M$ in $V$, there is a universe $W$ and embedding $h$

$$W \xrightarrow{h} V \xrightarrow{j} M$$

such that $j$ is the iterate of $h$.

Every embedding has already been iterated many times.
A consistency argument

Victoria Gitman and I have proved that, in suitably formalized form, these multiverse axioms are consistent.

Theorem (Gitman, Hamkins)

If ZFC is consistent, then the collection of countable computably-saturated models of ZFC satisfies all the multiverse axioms.

For example, to verify the Well-foundedness Mirage Axiom, we proved that for every countable computably-saturated model $M \models \text{ZFC}$, there is another such model $N$ such that $M$ exists inside $N$ and $N \models \text{"M has nonstandard natural numbers"}$. 
Mathematical Philosophy

The philosophical debate will likely not be settled by mathematical proof. But a philosophical view suggests mathematical questions and topics, and one measure of it is the value of the mathematics to which it has led.

The philosophical debate may be a proxy for: where should set theory go? Which mathematical questions should it consider?

So let me briefly describe some recent mathematics, undertaken from the multiverse perspective.

- Modal Logic of forcing. Upward-oriented, looking from a model of set theory to its forcing extensions.
- Set-theoretic geology. Downward-oriented, looking from a model of set theory down to its ground models.

This analysis engages pleasantly with various philosophical views on the nature of mathematical existence.
Affinity of Forcing & Modal Logic

Since a ground model has access, via names and the forcing relation, to the objects and truths of the forcing extension, the multiverse exhibits a natural modal nature.

- A sentence $\varphi$ is possible or forceable, written $\Diamond \varphi$, when it holds in a forcing extension.
- A sentence $\varphi$ is necessary, written $\Box \varphi$, when it holds in all forcing extensions.

Many set theorists habitually operate within the corresponding Kripke model, even if they wouldn’t describe it that way.
Maximality Principle

Q. Could the universe be completed, with respect to what is possibly necessary?

That is, could there be a model over which any possibly necessary assertion is already necessary?

The *Maximality Principle* is the scheme expressing

$$\diamondsuit \square \varphi \rightarrow \square \varphi$$

Theorem (Hamkins, also Väänänen, independently)

*The Maximality Principle is relatively consistent with ZFC.*

This work led to a consideration: what are the correct modal principles of forcing?
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This work led to a consideration: what are the correct modal principles of forcing?
Easy forcing validities

- **K**  
  \( \Box (\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi) \)

- **Dual**  
  \( \Box \neg \varphi \leftrightarrow \neg \Diamond \varphi \)

- **S**  
  \( \Box \varphi \rightarrow \varphi \)

- **4**  
  \( \Box \varphi \rightarrow \Box \Box \varphi \)

- **2**  
  \( \Diamond \Box \varphi \rightarrow \Box \Diamond \varphi \)

**Theorem**

Any S4.2 modal assertion is a valid principle of forcing.

**Question**

What are the valid principles of forcing?
Easy forcing validities

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- **.2** \( \Diamond \Box \varphi \rightarrow \Box \Diamond \varphi \)

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- **.2**: $\Diamond \Box \varphi \rightarrow \Box \Diamond \varphi$

**Theorem**

*Any S4.2 modal assertion is a valid principle of forcing.*

**Question**

*What are the valid principles of forcing?*
Beyond S4.2

\[5\] \(\Diamond \Box \varphi \rightarrow \varphi\)

\[\text{M}\] \(\Box \Diamond \varphi \rightarrow \Diamond \Box \varphi\)

\[\text{W5}\] \(\Diamond \Box \varphi \rightarrow (\varphi \rightarrow \Box \varphi)\)

\[.3\] \(\Diamond \varphi \land \Diamond \psi \rightarrow (\Diamond (\varphi \land \Diamond \psi) \lor \Diamond (\varphi \land \psi) \lor \Diamond (\psi \land \Diamond \varphi))\)

\[\text{Dm}\] \(\Box (\Box (\varphi \rightarrow \Box \varphi) \rightarrow \varphi) \rightarrow (\Diamond \Box \varphi \rightarrow \varphi)\)

\[\text{Grz}\] \(\Box (\Box (\varphi \rightarrow \Box \varphi) \rightarrow \varphi) \rightarrow \varphi\)

\[\text{Löb}\] \(\Box (\varphi \rightarrow \Box \varphi) \rightarrow \Box \varphi\)

\[\text{H}\] \(\varphi \rightarrow \Box (\Diamond \varphi \rightarrow \varphi)\)

It is a fun forcing exercise to show that these are invalid in some or all models of ZFC.
The resulting modal theories

\[
\begin{align*}
S5 &= S4 + 5 \\
S4W5 &= S4 + W5 \\
S4.3 &= S4 + .3 \\
S4.2.1 &= S4 + .2 + M \\
S4.2 &= S4 + .2 \\
S4.1 &= S4 + M \\
S4 &= K4 + S \\
Dm.2 &= S4.2 + Dm \\
Dm &= S4 + Dm \\
Grz &= K + Grz \\
GL &= K4 + Löb \\
K4H &= K4 + H \\
K4 &= K + 4 \\
K &= K + Dual
\end{align*}
\]
Valid principles of forcing

Theorem (Hamkins, Löwe)

If ZFC is consistent, then the ZFC-provably valid principles of forcing are exactly S4.2.

It was easy to show that S4.2 is valid.

The difficult part is to show that nothing else is valid.

If S4.2 ⊬ ϕ, we must provide ψ_i such that ϕ(ψ_0, ..., ψ_n) fails in some model of set theory.
Buttons and Switches

- A *switch* is a statement $\varphi$ such that both $\varphi$ and $\neg \varphi$ are necessarily possible.
- A *button* is a statement $\varphi$ such that $\varphi$ is (necessarily) possibly necessary.

**Fact.** Every statement in set theory is either a switch, a button or the negation of a button.

**Theorem**

*If $V = L$, then there is an infinite independent family of buttons and switches.*

Buttons: $b_n = “\mathcal{N}^L_n$ is collapsed ”

Switches: $s_m = “$ GCH holds at $\mathcal{N}_{\omega+m}$ ”
Structure of the argument

In order to show that non-S4.2 assertions are also not valid principles of forcing, we proceeded in two steps:

- First, we proved a new completeness theorem for S4.2 frames: the class of finite pre-lattices.
- Second, we proved that any finite Kripke model on such a frame is simulated via Boolean combinations of buttons and switches by forcing over a fixed model of set theory.

Step 2 made essential use of the Jankov-Fine modal expressions.

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Validities in a model

A fixed model $W$ of set theory can exhibit more than just the provable validities. Nevertheless,

**Theorem**

$$S4.2 \subseteq \text{Force}^W \subseteq S5.$$ 

Both endpoints occur for various $W$.

**Questions**

1. Can the validities of $W$ be strictly intermediate?
2. If $\varphi$ is valid for forcing over $W$, does it remain valid for forcing over all extensions of $W$?
3. Can a model of ZFC have an unpushed button, but not two independent buttons?
Surprising entry of large cardinals

Theorem. The following are equiconsistent:

1. $S5(R)$ is valid.
2. $S4W5(R)$ is valid for forcing.
3. $Dm(R)$ is valid for forcing.
4. There is a stationary proper class of inaccessible cardinals.

Theorem

1. *(Welch, Woodin)* If $S5(R)$ is valid in all forcing extensions (using $R$ of extension), then $AD^L(R)$.
2. *(Woodin)* If $AD_R + \Theta$ is regular, then it is consistent that $S5(R)$ is valid in all extensions.
Set-theoretic geology: A new perspective

Forcing is naturally viewed as a method of building *outer* as opposed to *inner* models of set theory.

Nevertheless, a simple switch in perspective allows us to view forcing as a method of producing inner models as well.

Namely, we look for how the universe $V$ might itself have arisen via forcing. Given $V$, we look for an inner model $W$ over which the universe is a forcing extension:

$$W \subseteq W[G] = V$$

This change in viewpoint results in the subject we call *set-theoretic geology*.

Haim Gaifman has pointed out (with humor) that this terminology presumes a stand on whether forcing is a human activity or a natural one...
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Digging for Grounds

A *ground* of the universe $V$ is a class $W$ over which the universe arises by forcing $V = W[G]$.

**Theorem (Laver, independently Woodin)**

*Every ground $W$ is a definable class in its forcing extensions $W[G]$, using parameters in $W$.*

**Definition (Hamkins, Reitz)**

The *Ground Axiom* is the assertion that the universe $V$ has no nontrivial grounds.

**Theorem (Reitz)**

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The Ground Axiom

The Ground Axiom holds in many canonical models of set theory: $L$, $L[0^#]$, $L[\mu]$, many instances of $K$.

**Question**

To what extent are the highly regular features of these models consequences of GA?

**Theorem (Reitz)**

*Not at all. Every model of ZFC has an extension, preserving any desired $V_\alpha$, which is a model of GA.*

**Theorem (Hamkins, Reitz, Woodin)**

*Every model of ZFC has an extension, preserving any desired $V_\alpha$, which is a model of GA + $V \neq \text{HOD}$.***
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Every model of ZFC has an extension, preserving any desired \( V_\alpha \), which is a model of \( \text{GA} + V \neq \text{HOD} \).
Bedrocks

$W$ is a *bedrock* of $V$ if it is a ground of $V$ and minimal with respect to the forcing extension relation.

Equivalently, $W$ is a bedrock of $V$ if it is a ground of $V$ and satisfies GA.

Open Question

Is the bedrock unique when it exists?

Theorem (Reitz)

*It is relatively consistent with ZFC that the universe $V$ has no bedrock model.*

Such models are *bottomless.*
The Mantle

We now carry the investigation deeper underground.

The principal new concept is the Mantle.

Definition

The Mantle $M$ is the intersection of all grounds.

The analysis engages with an interesting philosophical view: Ancient Paradise. This is the philosophical view that there is a highly regular core underlying the universe of set theory, an inner model obscured over the eons by the accumulating layers of debris heaped up by innumerable forcing constructions since the beginning of time. If we could sweep the accumulated material away, we should find an ancient paradise.

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Every model is a mantle

Although the *Ancient Paradise* philosophical view is highly appealing, our main theorem tends to refute it.

**Main Theorem (Fuchs, Hamkins, Reitz)**

Every model of ZFC is the mantle of another model of ZFC.

By sweeping away the accumulated sands of forcing, what we find is not a highly regular ancient core, but rather: an arbitrary model of set theory.
Downward directedness

The grounds of $V$ are *downward directed* if the intersection of any two of them contains a third.

$$W_t \subseteq W_r \cap W_s$$

In this case, there could not be two distinct bedrocks.

**Open Question**

Are the grounds downward directed? Downward set directed?

This is a fundamental open question about the nature of the multiverse.

In every model for which we can determined the answer, the answer is yes.
The Mantle under directedness

Theorem

1. If the grounds are downward directed, then the Mantle is constant across the grounds, and $M \models ZF$.
2. If they are downward set-directed, then $M \models ZFC$.

The Generic Mantle, denoted $gM$, is the intersection of all grounds of all forcing extensions of $V$. This includes all the grounds of $V$, and so $gM \subseteq M$.

Theorem

If the generic grounds are downward directed, then the grounds are dense below the generic multiverse, and so $M = gM$. 
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Theorem

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The generic multiverse

The *Generic Multiverse* of a model of set theory is the part of the multiverse reachable by forcing, the family of universes obtained by closing under forcing extensions and grounds.

There are various philosophical motivations to study the generic multiverse.

Woodin introduced the generic multiverse in order to reject a certain multiverse view of truth, the view that a statement is true when it is true in every model of the generic multiverse.

Our view is that the generic multiverse is a fundamental feature and should be a major focus of study.

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The generic multiverse of a model is the local neighborhood of that model in the multiverse.
The Generic Mantle

The generic Mantle is tightly connected with the generic multiverse.

**Theorem**

*The generic Mantle* $\mathcal{gM}$ *is invariant by forcing and* $\mathcal{gM} \models ZF$.

**Corollary**

The generic Mantle $\mathcal{gM}$ is constant across the generic multiverse. In fact, $\mathcal{gM}$ is the intersection of the generic multiverse.

The generic Mantle is the largest forcing-invariant class.

On this view, the generic Mantle is a canonical, fundamental feature of the generic multiverse.
The Generic HOD

HOD is the class of hereditarily ordinal definable sets.

\[ \text{HOD} \models ZFC \]

The generic HOD, introduced by Fuchs, is the intersection of all HODs of all forcing extensions.

\[ g\text{HOD} = \bigcap G \text{HOD}^{V[G]} \]

- \( g\text{HOD} \) is constant across the generic multiverse.
- The \( \text{HOD}^{V[G]} \) are downward set-directed. \( g\text{HOD} \models ZFC \).

\[ \text{HOD} \cup g\text{HOD} \subseteq gM \subseteq M \]
The Generic **HOD**

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The *generic* HOD, introduced by Fuchs, is the intersection of all HODs of all forcing extensions.

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- \( g\text{HOD} \) is constant across the generic multiverse.
- The \( \text{HOD}^{V[G]} \) are downward set-directed. \( g\text{HOD} \models \text{ZFC} \).

\[ g\text{HOD} \subseteq gM \subseteq M \]
Realizing $V$ as the Mantle

Theorem (Fuchs, Hamkins, Reitz)

If $V \models \text{ZFC}$, then there is a class extension $\overline{V}$ in which

$$V = M^{\overline{V}} = gM^{\overline{V}} = g\text{HOD}^{\overline{V}} = \text{HOD}^{\overline{V}}$$

In particular, as mentioned earlier, every model of $\text{ZFC}$ is the mantle and generic mantle of another model of $\text{ZFC}$.

It follows that we cannot expect to prove ANY regularity features about the mantle or the generic mantle.
Theorem (Fuchs, Hamkins, Reitz)

Other combinations are also possible.

1. Every model of set theory $V$ has an extension $\bar{V}$ with
   
   $V = M^{\bar{V}} = gM^{\bar{V}} = g\text{HOD}^{\bar{V}} = \text{HOD}^{\bar{V}}$

2. Every model of set theory $V$ has an extension $W$ with
   
   $V = M^W = gM^W = g\text{HOD}^W$ but $\text{HOD}^W = W$

3. Every model of set theory $V$ has an extension $U$ with
   
   $V = \text{HOD}^U = g\text{HOD}^U$ but $M^U = U$

4. Lastly, every $V$ has an extension $Y$ with
   
   $Y = \text{HOD}^Y = g\text{HOD}^Y = M^Y = gM^Y$
The Inner Mantles

When the Mantle $M$ is a model of $\text{ZFC}$, we may consider the Mantle of the Mantle, iterating to reveal the *inner Mantles*:

$$
M^1 = M \quad M^{\alpha+1} = M^{M^\alpha} \quad M^\lambda = \bigcap_{\alpha<\lambda} M^\alpha
$$

Continue as long as the model satisfies $\text{ZFC}$.

The *Outer Core* is reached if $M^\alpha$ has no grounds, $M^\alpha \models \text{ZFC} + \text{GA}$.

**Conjecture.** Every model of $\text{ZFC}$ is the $\alpha^{\text{th}}$ inner Mantle of another model, for arbitrary $\alpha \leq \text{ORD}$.

Philosophical view: ancient paradise?
Thank you.

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